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Metastates in Disordered Mean-Field Models II: The Superstates

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We continue to investigate the size dependence of disordered mean-field models with finite local spin space in more detail, illustrating the concept of "superstates" as recently proposed by Bovier and Gayrard. We discuss various notions of convergence for the behavior of the paths $(t \to \mu_{[tN]}(\eta))_{t \in (0,1]}$ in the thermodynamic limit $N \uparrow \infty$. Here $\mu_n(\eta)$ is the Gibbs measure in the finite volume $\{1,..., n\}$ and η is the disorder variable. In particular we prove refined convergence statements in our concrete examples, the Hopfield model with finitely many patterns (having continuous paths) and the Curie–Weiss random-field Ising model (having singular paths).

KEY WORDS: Disordered systems; size dependence; random Gibbs states; metastates; superstates; mean-field models; Hopfield model; random field model.

1. INTRODUCTION

In a previous paper [K1] we discussed examples of the behavior of metastates (introduced by Newman and Stein) for random mean field spin models with quadratic interaction and finite state space. For a motivation for metastates in the context of spin glasses and the analysis of chaotic size dependence in random spin models we refer the reader to [N], [NS1]–[NS5], [K1]. Let us however remind the reader of the basic underlying idea: The metastate is a probability measure on the states of the system that gives, informally speaking, the likelihood of finding the system in a particular state if we choose a large volume "at random."

Thus, the metastate says more about the large volume behavior of a system than the set of possible *limit points* of states for subsequences of

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volumes; it contains also the important knowledge of the "weights" for finding them in a large volume.

If there is only one state for the system or the boundary conditions preselect one particular state, the system will approach it for large volumes; so the metastate will be trivial (i.e., a Dirac measure on this state). Also, in translation invariant systems, even in the multiphase region, the size-dependence is usually simple. Thus a metastate-description is not needed in these cases. For quenched disordered systems in the multi-phase region whose symmetry is *randomly broken* however, the size dependence can be complicated ("chaotic"); if this is the case (as it is already in our simple examples), the metastate is the natural object to describe the large volume behavior. The actual probabilistic construction of metastates then either involves sampling of the system over sequences of volumes (following Newman/Stein) or averaging "over the couplings at infinity" (Aizenman/Wehr [AW]).

Moreover, for some aspects of the size dependence of quenched random systems and the statistics of observables along sequences of volumes, the concept of metastates needs some generalization, to be explained below. The present follow-up note continues the discussion of the random mean field models with quadratic interaction considered in [K1] in this light. We assume that the reader is familiar with this paper and make use of the results proved therein.

To explain the kind of questions we are interested in here, let us give the following simple concrete example. We consider a spin-system in the multi-phase region whose symmetry is randomly broken. Let us fix two sites x_1, x_2 and consider $\mu_n(\eta)(\sigma_{x_1}\sigma_{x_2})$, the correlation between two spins w.r.t. the finite volume Gibbs measures $\mu_n(\eta)$ observed along a sequence of volumes labeled by $n = 1, 2,...,$ for fixed realization of the quenched random variable η. Assume that we know the realization of the random variables describing the quenched randomness "inside" a fixed *finite* volume containing the sites x_1, x_2 (but not in the infinite volume.) We might ask ourselves e.g., what will be the behavior of the maximum of $\mu_n(\sigma_{x_1}\sigma_{x_2})$ taken over volumes $n = N/2$, $N/2 + 1,...,$ N when N goes to infinity? What is the distribution of this quantity if we randomly choose a very large N?

To answer this question we need an object to describe asymptotic size dependence that is even more general than the limiting distributions of metastates constructed in [K1]. Indeed, one aspect of this is that we have to deal with distributions of *paths of volumes* $t \mapsto \mu_{[tN]}(\xi)$ (Here $[s]$ is the largest integer less or equal to s). Probabilistically speaking, this means that we want to strengthen a limit theorem to an invariance principle (functional limit theorem). The other aspect is that we look at these distributions while *fixing* the random variables describing the quenched disorder in *any finite volume*. To formulate in general limit statements arising in this situation we will need the notion of convergence in "locally conditioned law" (or generalized Aizenman–Wehr construction [AW]), that seems to be appropriate to describe the convergence of many objects in disordered systems. It is stronger than "convergence in law." When applied directly to paths of volumes $t \mapsto \mu_{[tN]}(\xi)$, this will lead us to the notion of "superstates", that was proposed by Bovier and Gayrard in [BG3]. We will then also have to discuss the right notion of convergence of paths of Gibbs measures, as far as the t- (meaning: volume-) dependence is concerned. This question depends on the specific model, as we will illustrate in our two main examples. Our point is here again to develop the correct framework in a rigorous way and illustrate concepts that are of general interest in disordered systems in a simple situation (that has some pedagogical value). In answer to the above question about maxima of correlations we will prove that e.g., in the concrete example of the finite pattern Hopfield model there is indeed a non-degenerate limiting distribution that is explicitly computable. (In more realistic models explicit computations will of course hardly ever be possible.)

Possible further extensions in the study of chaotic size dependence are of course to consider lattice systems and/or systems with infinitely many extremal Gibbs states. Let us only remark here that a new phenomenon can appear when one is dealing with infinitely many pure states, in that we might encounter either *recurrence* or *transience* of Gibbs measures ([K3]). By recurrence we mean that, for each pure random infinite volume Gibbs measure, there exist subsequences of volumes for which the system is "close" to it. Recurrence was trivially true for the mean field systems in [K1] having *finitely many* extremal Gibbs states. When one does not deal with the local topology but imposes a stronger notion of "proximity", or in the case of a non-compact spin space, it is however not to be taken for granted anymore.

Very recently the conditioned (Aizenman–Wehr) metastate of the standard Hopfield model, conditioned on one pattern, in the replica symmetric regime has been constructed, using the local topology [BG3]. It provides a very interesting example of a metastate supported on a continuum of extremal Gibbs measures. Another explicit new example of metastates in models with finitely many pure phases are mean field models that are randomly perturbed by a small SK-term, considered in [To].

The organization of the paper is as follows. In Chapter 2 we describe the notion of convergence in locally conditioned law. In Chapter 3 we remind the reader of the mean field models treated in [K1] and prove approximation statements useful for superstates. In Chapter 4 we discuss the Curie–Weiss Random Field Ising Model (see Theorem 1) and, more interestingly in this context, the finite pattern Hopfield Model (see Theorem 2).

2. CONVERGENCE IN LOCALLY CONDITIONED LAW—THE GENERAL AIZENMAN-WEHR—CONSTRUCTION

It is important in disordered systems to distinguish between the *local* influences of the randomness on the behavior of the system and the "*global*" ones. The conceptual interest of the Aizenman–Wehr metastate lies in the fact that it provides a mathematical framework for limiting statements of these questions. Indeed, it is a probability distribution on the Gibbs measures of a disordered system that, loosely speaking, describes the behavior of Gibbs-measures given the *local* knowledge of the randomness, but subjected to the fluctuations of the randomness "*at infinity*." It is now interesting to investigate in the same way also possibly different objects than the Gibbs-measures themselves w.r.t. their behavior as random elements. To describe various convergence statements arising naturally in disordered systems in a unified way it is useful to take an abstract point of view and perform the Aizenman–Wehr construction (see [AW], [N], [K1]) in the following generalized manner. This will lead in a direct way to what [BG3] call superstates. It also helps to keep the language reasonably simple, avoiding to speak of hierarchies of metastates.

In disordered systems we often encounter sequences of maps $\phi_N : \mathscr{H} \to \mathscr{E}$, given in some implicit way, from the space $\mathscr{H}$ of random variables describing the disorder to some "complicated" Polish space $\mathscr{E}$. We assume as always that $\mathscr{H}$ is a countable product of Polish spaces over a lattice. A natural example for ϕ_N is the map that associates to a realization of the disorder η a certain (finite volume) Gibbs measure $\mu_N(\eta)$. Later we will even consider the cases where $\mathscr{E}$ is the space of (empirical) distributions of Gibbs measures or the space of paths of Gibbs measures. Now, looked upon from an abstract point of view, the only structure needed for the AW-construction is the notion of a "local function of the disorder variables"; the actual image space ϕ_N takes values in plays no role. Thus, we can make the following more general definition that extends the notion of convergence in law. Thereby we also introduce notation that is suitable for later use.

Definition (Convergence in lc-law). Let $\phi_N : \mathscr{H} \to \mathscr{E}$ be a sequence of measurable maps of the disorder space $\mathscr{H}$ into some Polish space $\mathscr{E}$. Let g be another random variable (with values in some other Polish space), independent of η, and $\phi : (\eta, g) \mapsto \phi(\eta, g)$ a map, measurable w.r.t. the product topology.

We say that $\phi_N(\eta)$ converges in locally conditioned law to $\phi(\eta, g)$ if, for each finite Λ, we have, for each bounded continuous function $F: \mathscr{E} \times \mathscr{H}_\Lambda \to \mathbb{R}$,

$$\lim_{N \uparrow \infty} \mathbb{E}[F(\phi_N(\eta), \eta_\Lambda)] = \mathbb{E}\mathbb{E}_g F(\phi(\eta, g), \eta_\Lambda) \tag{2.1}$$

Here $\mathscr{H}_\Lambda$ denotes the product of the local disorder spaces over sites in Λ.

Later we will also use the notation $\lim_{N \uparrow \infty} \phi_N(\eta) =^{\text{lc-law}} \lim_{N \uparrow \infty} \bar{\phi}_N(\eta, g)$ if

$$\lim_{N \uparrow \infty} \mathbb{E}[F(\phi_N(\eta), \eta_\Lambda)] = \lim_{N \uparrow \infty} \mathbb{E}\mathbb{E}_g[F(\bar{\phi}_N(\eta, g), \eta_\Lambda)] \tag{2.2}$$

holds whenever the r.h.s. exists.

The requirement (2.1) means that convergence in law still holds if any finite number of r.v.'s is kept fixed. Indeed, note that $\lim_{N \uparrow \infty} \phi_N(\eta) =^{\text{lc-law}} \phi(\eta, g)$ implies trivially convergence in law, $\lim_{N \uparrow \infty} \phi_N(\eta) =^{\text{law}} \phi(\eta, g)$. (We need to check (2.1) only for the smaller set of F's that do not depend on the last variable.) The opposite is of course not true: Take as an example η_i's i.i.d. non-constant random variables. Take $\mathscr{E} = \mathbb{R}^2$ and define the map $\phi_N^1(\eta) = (\eta_1, \eta_2)$ for N even, and $\phi_N^1(\eta) = (\eta_2, \eta_1)$ for N odd. Then ϕ_N^1 converges trivially in law to a pair of independent versions $(\tilde{\eta}_1, \tilde{\eta}_2)$, but does *not* converge in lc-law. (Consider the function $F(\phi, \eta) = 1_{\phi_1 = \eta_1}$ in (2.1).)

Our next example is a simplified version of the process arising in the analysis of the size dependence of the finite pattern Hopfield model in [K1] and was also used as an example in [BG3]. Take $\phi_N^2(\eta) = (\eta_1, 1/\sqrt{N} \sum_{i=1,\dots,N} \eta_i)$ where η_i are (e.g.) i.i.d. centered variables with finite second moment. Using the central limit theorem, it is then not difficult to see that we have $\lim_{N \uparrow \infty} \phi_N^2(\eta) =^{\text{lc-law}} (\eta_1, g)$ where g is a Gaussian. This is in particular true, if the η_i's are Gaussian. Thus, in the case of Gaussians, the sequences ϕ_N^1 and ϕ_N^2 even have the same limit in law, but different behavior in lc-law.

This example mimics the situation in disordered systems in the following respect: The fixing of η_1 (or any finite number of η_i's) corresponds to the "knowledge of the disorder variables in some finite volume." A "global quantity" like $1/\sqrt{N} \sum_{i=1,\dots,N} \eta_i$ should be thought of decisive for the "global decision" of the system of which of the phases to prefer. The above simple limit statement then says that any finite local knowledge does not influence the asymptotic "global behavior."

We will now consider more interesting spaces $\mathscr{E}$ that arise from Gibbs measures on a spin space Ω whose disorder variables are the ηs. Let us, as usual, assume that the spin space Ω is a countable product over a finite set S, equipped with the product topology. By $\mathscr{P}(\Omega)$ we denote the set of probability measures on Ω (the set of *states*), equipped with the weak topology, which makes it Polish. The set of *metastates* is the set of probability measures on the states, $\mathscr{P}(\mathscr{P}(\Omega))$, equipped with the weak topology inherited from $\mathscr{P}(\Omega)$.

Now, for a sequence $\phi_N(\eta)$ given by a sequence of finite volume Gibbs measures $\mu_N(\eta) \in \mathscr{E} = \mathscr{P}(\Omega)$ the statement $\lim_{N \uparrow \infty} \mu_N(\eta) =^{\text{lc-law}} \mu(\eta, g)$ is just a different notation to describe the AW-metastate. Indeed, it says that the distribution of the Gibbs measures μ under the AW-metastate $\bar{\kappa}(\eta)$ is induced by the distribution of g under the measurable map $\mu(\eta, g)$ for fixed η.

To get something new, let us consider the sequence of *empirical metastates* $\kappa_N(\eta) = 1/N \sum_{n=1}^{N} \delta_{\mu_n(\eta)} \in \mathscr{P}(\mathscr{P}(\Omega))$ that is associated to a sequence of finite volume Gibbs measures $\mu_n(\eta)$. The statement $\lim_{N \uparrow \infty} \kappa_N(\eta) =^{\text{lc-law}} \kappa(\eta, g)$ with a given map $\kappa(\eta, g)$ could then be rephrased as "$\kappa(\eta, g)$ describes the distribution of metastates (induced by g) under an Aizenman–Wehr metastate for the empirical distributions." In fact, this gives logically a stronger statement than simple convergence in law, in the way it was proved in our examples in [K1]. We will however see later that convergence in law proved in those examples carries over in a direct way to convergence in lc-law. Extending the empirical distribution for Gibbs measures we can also consider generalized empirical averages of the form $\phi_N(\eta) = \sum_{n=1}^{N} G(\mu_n(\eta), n/N)$, where G is a continuous function on $\mathscr{P}(\Omega) \times [0, 1]$.

Finally we can look directly at the paths $\phi_N(\eta) = (t \mapsto \mu_{Nt}(\eta))_{0 \leqslant t \leqslant 1}$. Here we write μ_s for non-integer s for the element in $\mathscr{P}(\Omega)$ obtained by linear interpolation between $\mu_{[s]}$ and $\mu_{[s+1]}$. ($[s]$ denotes the largest integer less or equal to s.) The distribution of the limit in lc-law was called the *superstates* in [BG3]. A first step to control this object is at any rate to look at the finite dimensional marginals, that is $\phi_N(\eta) = (\mu_{[t_1 N]}(\eta), ..., \mu_{[t_k N]}(\eta))$ for fixed $0 < t_1 < t_2 < \cdots < t_k \leqslant 1$. If we are for the moment ignoring topological subtleties (that can however be important), we formally get back the empirical distribution from the whole path behavior. Whether we can expect in the limit continuous trajectories (as a function of t), jump processes, or more singular trajectories has then to be investigated in the specific model. Also the notion of convergence has consequently to be adapted to the model. When we come to our examples, we will see that the finite pattern Hopfield model has asymptotically continuous trajectories, allowing for convergence in a metric that is uniform in t, in contrast to the singular trajectories in the Curie–Weiss Random Field Ising model.

3. MEAN FIELD MODELS WITH QUADRATIC INTERACTION

In [K1] we treated certain quadratic random mean field models and stated some approximation properties for them that were good enough to imply in a direct way results for metastates in terms of the asymptotic weights. These criteria were then shown to hold for our main examples, the CWRFIM and the finite pattern Hopfield model.

The very same criteria are also useful to treat the superstates, as we will see in Proposition 1. In cases where we want to obtain convergence statements for a metric that is *uniform* in the rescaled volume label t we have to exclude that the trajectories become "increasingly rough" with large N. This means that the finite-volume Gibbs-measures corresponding to (sufficiently large) volumes that differ only in a small fraction of sites are similar. While this is also true for "most" of such pairs of Gibbs-measures in the CWRFIM, it does not hold for all of them, leading to asymptotically singular trajectories (see Theorem 1). We will show below in Lemma 1 how we can decide these "tightness questions" on the level of the asymptotic weights.

Let us recall the models we treated in [K1] and remind the reader of some of the notation used therein. The spins $\sigma = (\sigma_i)_{i=1,2,...}$ taking values in the infinite product space $\Omega = S^{\mathbb{N}}$ over a finite[3] set S have an a priori distribution according to a (possibly random) product measure $\mu^0(\eta)[\sigma_\Lambda = \omega_\Lambda] = \prod_{i\in\Lambda}\mu_i^0(\eta_i)[\sigma_i = \omega_i]$ where η_i, $i \in \mathbb{N}$ are random variables. The quadratic mean field energy function is of the form

$$E_N(\sigma, \eta) = -\frac{N}{2}\bar{m}_N(\sigma, \eta)^2 \equiv -\frac{N}{2}\sum_{\nu=1}^{M}\bar{m}_N^\nu(\sigma, \eta)^2 \tag{3.1}$$

where the *order parameter* $\bar{m}_N$ is defined by the empirical average $\bar{m}_N(\sigma, \eta) := 1/N\sum_{i=1}^N m(\sigma_i, \eta_i)$ with some bounded continuous map $(\sigma_1, \eta_1) \mapsto m(\sigma_1, \eta_1)$ taking values in $\mathbb{R}^M$. The finite volume Gibbs measures are then

$$\mu_N(\eta)[\sigma = \omega] := \frac{\exp(-\beta E_N(\omega, \eta))}{Norm.}\mu^0(\eta)[\sigma = \omega] \tag{3.2}$$

Our two main examples are the following. Both have Ising spins $\sigma_i \in \{-1, 1\}$.

Curie–Weiss Random Field Ising Model (CWRFIM). The a priori measures $\mu_i(\eta_i)[\sigma_i = \pm 1] = (e^{\pm\beta\hat{\varepsilon}\eta_i})/(2\cosh(\beta\hat{\varepsilon}\eta_i))$ are random,

[3] We stick to the models with finite local state-space of [K1]; our main specific examples are even Ising models.

where, for simplicity, η_i are i.i.d. Bernoulli variables, taking the values ± 1 with equal probability. The order parameter is given by $m((\sigma_1, \eta_1) = \sigma_1$, $\hat{\varepsilon} > 0$ is the magnitude of the disorder.

***M*-Pattern Hopfield Model.** The a priori measures for the spins are symmetric Bernoulli. We call the random variables in this case ξ instead of η. $(\xi_i^\mu)_{i=1,2,\dots;\ \mu=1,\dots,M} \equiv (\xi_i)_{i=1,2,\dots}$ are i.i.d. (for different i, μ) with $\mathbb{P}[\xi_i^\mu = \pm 1] = \frac{1}{2}$. The order parameter is given by $m(\sigma_1, \xi_1) = \sigma_1 \xi_1 \in \{1, -1\}^M$. The empirical mean $\bar{m}_N(\sigma, \xi)$ is the *overlap vector*.

Going back to the general case we also have to remind the reader of the Hubbard–Stratonovich measures

$$\tilde{\mu}_N(\eta)(dm) = \frac{\exp(-\beta N \Phi_N(m, \eta))\, dm}{\int_{\mathbb{R}} dm' \exp(-\beta N \Phi_N(m', \eta))} \tag{3.3}$$

given by the function

$$\Phi_N(m, \eta) = \frac{m^2}{2} - \frac{1}{N} \sum_{1 \leqslant i \leqslant N} L(m, \eta_i) \tag{3.4}$$

where $L(t, \eta_i) = 1/\beta \log \int \mu_i^0(\eta_i)(d\sigma_i) \exp(\beta t \cdot m(\sigma_i, \eta_i))$ is the logarithmic moment generating function of the tilted measures

$$\mu_i^0(t, \eta_i)[\sigma_i = \omega_i] = \frac{\exp(\beta t \cdot m(\omega_i, \eta_i))}{\exp(\beta L(t, \eta_i))} \mu_i^0(\eta_i)[\sigma_i = \omega_i] \tag{3.5}$$

The infinite product measures $\mu_\infty^0(m, \eta)[\sigma_\Lambda = \omega_\Lambda] = \prod_{i \in \Lambda} \mu_i^0(m, \eta_i)[\sigma_i = \omega_i]$ play the role of the *infinite volume Gibbs measures* when m is an element of the set

$$\mathcal{M} = \left\{ m,\ \mathbb{E}[\Phi_N(m, \eta)] = \min_{m'} \mathbb{E}[\Phi_N(m', \eta)] \right\} \tag{3.6}$$

which labels the pure phases according to their "magnetization."

The problem of the analysis of size dependence is to make rigorous an approximation of the form $\mu_N(\eta) \approx \sum_{m \in \mathcal{M}} p_N^m(\eta)\, \mu_\infty^0(m, \eta)$ where $(p_N^m(\eta))_{m \in \mathcal{M}}$ is a sequence of random probability vectors, indexed by N, that we will refer to as "the asymptotic form of the weights." This was done in [K1] where we defined the following approximation property that was useful for the metastate analysis and will also be useful here. It was then shown to hold for our main examples by explicit saddle point analysis. Assume that we are given a sequence of "nice" subsets $\mathcal{H}(N) \subset \mathcal{H}$ of the disorder space. We write $\underline{\mathcal{H}} = \liminf_N \mathcal{H}(N)$ for the set of η's that are contained in all $\mathscr{H}(N)$ with possibly finitely many exceptions. We also use the bigger "nice" set $\mathscr{H}' = \{\eta, \lim_{N\uparrow\infty} 1/N \sum_{n=1}^{N} 1_{\eta\in\mathscr{H}(n)^c} = 0\}$. We need

Property 1. We say that $\tilde{\mu}_N(\eta)$ becomes close to the probability vector $(p_N^m(\eta))_{m\in\mathscr{M}}$ along the regular sets $\mathscr{H}(N)$ if, for all $\eta\in\mathscr{H}$, for all $m\in\mathscr{M}$, $\lim_{N\uparrow\infty}(\tilde{\mu}_N(\eta)[B_{\rho_N}(m)] - p_N^m(\eta)) = 0$ for a decreasing sequence of radii $\rho_N \downarrow 0$.

We will also assume the following piece of information on the asymptotic form of the weight (shown to hold in our specific cases in [K1]) that we call here

Property 2. For all $\eta\in\mathscr{H}$, for all finite $V\subset\mathbb{N}$, we have $\lim_{N\uparrow\infty} \sup_{\tilde{\eta}_V} \|p_N(\eta) - p_N(\eta+\tilde{\eta}_V)\| = 0$ where $\tilde{\eta}_V$ is a local perturbation in the finite volume V s.t. $\eta_V + \tilde{\eta}_V$ lies in the support of the distribution $\mathbb{P}$.

Denoting again the empirical metastate $\kappa_N(\eta) = 1/N \sum_{n=1}^{N} \delta_{\mu_n(\eta)}$ we may now get in this situation in extension of Proposition 1 of [K1]

Proposition 1. Suppose that we are given a quadratic random mean field model of the above type that satisfies the properties 1 and 2. Let η' denote a copy of disorder variables, independent of η.

(i) Assume that $\mathbb{P}[\mathscr{H}'] = 1$. Then we have

$$\lim_{N\uparrow\infty} \kappa_N(\eta) =^{\text{lc-law}} \lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{\sum_{m\in\mathscr{M}} p_n^m(\eta')\,\mu_\infty^0(m,\eta)} \tag{3.7}$$

and, more generally, for each bounded continuous $G\colon \mathscr{P}(\Omega)\times[0,1]\to\mathbb{R}^k$,

$$\lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^{N} G\left(\mu_n(\eta), \frac{n}{N}\right) =^{\text{lc-law}} \lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^{N} G\left(\sum_{m\in\mathscr{M}} p_n^m(\eta')\,\mu_\infty^0(m,\eta), \frac{n}{N}\right) \tag{3.8}$$

(ii) Assume that $\lim_{N\uparrow\infty}\mathbb{P}[\mathscr{H}(N)] = 1$. Then, for all $0 < t_1 < t_2 < \cdots < t_k \leqslant 1$,

$$\begin{aligned} &\lim_{N\uparrow\infty} (\mu_{[t_1 N]}(\eta), \ldots, \mu_{[t_k N]}(\eta)) \\ &\quad =^{\text{lc-law}} \lim_{N\uparrow\infty} \sum_{m\in\mathscr{M}} (p_{[t_1 N]}^m(\eta'), \ldots, p_{[t_k N]}^m(\eta'))\,\mu_\infty^0(m,\eta) \end{aligned} \tag{3.9}$$

Remark. Note that we need for (ii) that t_1 is strictly bigger than 0. As explained in Chapter 2, the ηs are *the same* on both sides of the equations (in contrast to [K1])!

The proof uses essentially arguments as in Proposition 1 from [K1]; we will postpone it to the end of this chapter. To proceed further let us assume now that we are in a situation where we have exhibited a proper continuous time process and proved the limit for paths of Gibbs measures in the sense of convergence of finite dimensional marginals. Let us now ask for convergence in the stronger sense of a metric that is *uniform in* t, thereby forcing us to consider tightness questions. Indeed, the experience from [K1] provides us with an example where we expect "asymptotically continuous" paths, namely the finite pattern Standard Hopfield model. It means that, for large volumes $N' \geqslant N$ differing only in a small fraction δ of sites, the Gibbs measures are "close," uniformly in N. However, one can expect the paths to be continuous only on the *open* interval $(0, 1]$ which leads us to work in fact with the family of metrics given by

$$d_\varepsilon(t \mapsto \mu_t, t \mapsto \mu'_t) := \sup_{t \in [\varepsilon, 1]} d(\mu_t, \mu'_t) \tag{3.10}$$

where, to be definite, we now choose the metric on $\mathscr{P}(\Omega)$

$$d(\mu, \mu') = \sum_{N=1}^{\infty} 2^{-N} \sum_{\omega_{\Lambda_N} \in \Omega_{\Lambda_N}} |\mu[\sigma_{\Lambda_N} = \omega_{\Lambda_N}] - \mu'[\sigma_{\Lambda_N} = \omega_{\Lambda_N}]| \tag{3.11}$$

with the volumes $\Lambda_N = \{1,..., N\}$. Then we consider the weak limit on the space of paths $(t \mapsto \mu_t)_{t \in [\varepsilon_1, 1]}$ equipped with the metric d_{ε_1}, simultaneously for any fixed $\varepsilon_1 > 0$.

It is well known that for such limit to hold we need 1) convergence of finite dimensional marginals and 2) that the sequence of paths is *tight*, meaning precisely that for each fixed $\varepsilon > 0$.

$$\lim_{\delta \downarrow 0} \sup_{n=1,2,...} \mathbb{P}\left[\sup_{\substack{\varepsilon_1 \leqslant s, t < 1 \\ |s-t| \leqslant \delta}} d(\mu_{ns}(\xi), \mu_{nt}(\xi)) \geqslant \varepsilon \right] = 0 \tag{3.12}$$

(see, e.g., [Po], Th. V.3, p. 92). It suffices here of course to consider integer volumes since $\sup_{s, s' : A \leqslant s, s' \leqslant B} d(\mu_s, \mu'_{s'}) \leqslant \sup_{n, n' \in \mathbb{N} : [A] \leqslant n, n' \leqslant [B]+1} d(\mu_n, \mu'_{n'})$.

Let us point out that tightness is *not* trivially true in the mean field models we consider (in the concrete example of the CWRFIM it is also actually false). Indeed, to treat the difference between the weights in volumes N, N' with $N \leqslant N' \leqslant (1+\delta) N$ with a priori estimates we could try to consider just the supremum over N, N' over the arguments of the exponentials occurring in the $\tilde{\mu}$-integrals, i.e.,

$$\sup_{N' : N \leqslant N' \leqslant (1+\delta) N} |N\Phi_N(m, \eta) - N'\Phi_{N'}(m, \eta)|$$

$$= \sup_{N' : N \leqslant N' \leqslant (1+\delta) N} \left| \sum_{i : N < i \leqslant N'} L(m, \eta_i) \right| \tag{3.13}$$

with m varying in a ball around a specific minimizer. For *fixed* m this quantity will or course be of the order of $\sqrt{\delta N}$ which diverges with N. This simple consideration however does not take into account that the "true minimizers" might give better estimates, as it is the case in the Hopfield model.

The next probabilistic lemma shows, that assuming a nice a.s. approximation for the Gibbs measures it suffices already to check tightness on the level of the *asymptotic weights.*

Lemma 1. Assume that $\lim_{N\uparrow\infty} d(\mu_N(\eta), \sum_{m\in\mathcal{M}} p_N^m(\eta)\, \mu_\infty^0(m, \eta)) = 0$, a.s. Then the condition for the asymptotic weights $p_N(\eta)$ given by

$$\limsup_{\delta \downarrow 0} \sup_{\bar{N}} \mathbb{P}\left[\sup_{\substack{N, N' : \varepsilon_1 \bar{N} \leqslant N < N' \leqslant N + \delta\bar{N} \\ N' \leqslant \bar{N}}} \|p_N(\eta) - p_{N'}(\eta)\| \geqslant \varepsilon \right] = 0 \tag{3.14}$$

for any $\varepsilon > 0$, implies the tightness condition for the finite volume Gibbs measure (3.12) where $\|\cdot\|$, any norm on $\mathbb{R}^M$ and $\varepsilon_1 > 0$ fixed.

Remark. The lemma is useful because, assuming property 1 and $\mathbb{P}[\mathscr{H}] = 1$, it was shown in [K1] that the first hypothesis of the lemma holds. So, in Chapter 4, we don't need any new refined explicit saddle point estimates and can obtain results about the superstates as corollaries of our previous approximation results.

Proof. It is useful to make explicit the map T_η that associates to an element p in $\mathscr{S}$, the simplex of $|\mathcal{M}|$-dimensional probability vectors, the infinite volume Gibbs measure

$$T_\eta(p) = \sum_{m\in\mathcal{M}} p^m \mu_\infty^0(m, \eta) \tag{3.15}$$

We have $d(T_\eta(p), T_\eta(\tilde{p})) \leqslant 2 \sum_{m\in\mathcal{M}} |p^m - \tilde{p}^m|$, so T_η is Lipshitz if we put a distance $d(p, \tilde{p}) := \|p - \tilde{p}\|$ on $\mathscr{S}$. Let us now write in short for the approximated measures $\mu'_N(\eta) = \sum_{m\in\mathcal{M}} p_N^m(\eta)\, \mu_\infty^0(m, \eta) = T_\eta(p_N(\eta))$. We have

$$\sup_{\bar N \geqslant N_0} \mathbb{P}\left[\sup_{\substack{N, N' : \varepsilon_1 \bar N \leqslant N < N' \leqslant N + \delta \bar N \\ N' \leqslant \bar N}} d(\mu_N(\eta), \mu_{N'}(\eta)) \geqslant \varepsilon \right]$$

$$\leqslant \sup_{\bar N \geqslant N_0} \mathbb{P}\left[\sup_{\substack{N, N' : \varepsilon_1 \bar N \leqslant N < N' \leqslant N + \delta \bar N \\ N' \leqslant \bar N}} d(\mu'_N(\eta), \mu'_{N'}(\eta)) \geqslant \frac{\varepsilon}{2} \right]$$

$$+ \mathbb{P}\left[2 \sup_{N : N \geqslant \varepsilon_1 N_0} d(\mu_N(\eta), \mu'_N(\eta)) \geqslant \frac{\varepsilon}{2} \right] \tag{3.16}$$

From the a.s. convergence in the hypothesis follows in particular (from dominated convergence) that the second term on the r.h.s. converges to zero with $N_0 \uparrow \infty$ since ε_1 is strictly positive. For fixed small $\eta > 0$ we can thus choose N_0 s.t. this term is bounded from above by η. Thus we get

$$\limsup_{\delta \downarrow 0} \sup_{\bar N} \mathbb{P}\left[\sup_{\substack{N, N' : \varepsilon_1 \bar N \leqslant N < N' \leqslant N + \delta \bar N \\ N' \leqslant \bar N}} d(\mu_N(\eta), \mu_{N'}(\eta)) \geqslant \varepsilon \right]$$

$$\leqslant \limsup_{\delta \downarrow 0} \sup_{\bar N : \bar N < N_0} \mathbb{P}\left[\sup_{\substack{N, N' : \varepsilon_1 \bar N \leqslant N < N' \leqslant N + \delta \bar N \\ N' \leqslant \bar N}} d(\mu_N(\eta), \mu_{N'}(\eta)) \geqslant \frac{\varepsilon}{2} \right]$$

$$+ \limsup_{\delta \downarrow 0} \sup_{\bar N \geqslant N_0} \mathbb{P}\left[\sup_{\substack{N, N' : \varepsilon_1 \bar N \leqslant N < N' \leqslant N + \delta \bar N \\ N' \leqslant \bar N}} d(\mu'_N(\eta), \mu'_{N'}(\eta)) \geqslant \frac{\varepsilon}{2} \right] + \eta \tag{3.17}$$

The first limsup on r.h.s. is zero, trivially. Since η is arbitrary it suffices to show that for fixed $\varepsilon > 0$, for any N_0 we have that the second limsup on the l.h.s. is zero. But this follows from the hypothesis by the Lipshitz continuity of T_η. ▮

We finish this chapter with the

Proof of Proposition 1. Remember that functions of the type $G(\mu) = \tilde G(\mu(f_1),..., \mu(f_l))$ are dense in the continuous functions on $\mathscr{P}(\Omega)$ w.r.t. sup-norm, where the f_i's are local and $\tilde G$ is continuous on $\mathbb{R}^l$ (see [AW]). The functions of the type $F(\kappa) = \tilde F(\kappa(G_1),..., \kappa(G_m))$ are dense in the continuous functions on $\mathscr{P}(\mathscr{P}(\Omega))$, where the functions $G_j(\mu)$ are of the above form and $\tilde F$ is continuous on $\mathbb{R}^m$. To show (i), we only need to look at

$$\lim_{N \uparrow \infty} \mathbb{E} F(\kappa_N(\eta); \eta) = \mathbb{E} \tilde F\left(\frac{1}{N} \sum_{n=1}^{N} G(\mu_n(\eta)); \eta \right) \tag{3.18}$$

where $G: \mathcal{P}(\Omega) \to \mathbb{R}^k$ is a continuous function on the states and the dependence of $\tilde{F}$ on η in the last argument is local. This shows in particular that the second statement in (i) (which allows for volume dependent G's) is more general.

To show the locally conditioned convergence in the case of possibly volume dependent G's we have to show that, for any continuous $F: \mathbb{R}^k \times \mathcal{P}(\Omega) \to \mathbb{R}$, we have that

$$\lim_{N \uparrow \infty} \mathbb{E} F\left(\frac{1}{N} \sum_{n=1}^{N} G\left(\mu_n(\eta), \frac{n}{N}\right); \eta\right) = \lim_{N \uparrow \infty} \mathbb{E}\mathbb{E}' F\left(\frac{1}{N} \sum_{n=1}^{N} G\left(\sum_{m \in \mathcal{M}} p_n^m(\eta')\, \mu_\infty^0(m, \eta), \frac{n}{N}\right); \eta\right) \tag{3.19}$$

where $\mathbb{E}'$ denotes expectation w.r.t the independent variable η'. For the first argument of the function F we can write

$$\frac{1}{N} \sum_{n=1}^{N} G\left(\mu_n(\eta), \frac{n}{N}\right) 1_{\mathcal{H}(n)} + \mathcal{O}\left(\|G\|_\infty \frac{1}{N} \sum_{n=1}^{N} 1_{\mathcal{H}(n)^c}\right) \tag{3.20}$$

Recall now that from property 1 follows that $d(\mu_n(\eta), \sum_{m \in \mathcal{M}} p_n^m(\eta)\, \mu_\infty^0(m, \eta))$ $1_{\mathcal{H}(n)} \to 0$ with $n \uparrow \infty$ (see Lemma 2 [K1]), where d is a metric on $\mathcal{P}(\Omega)$ for local convergence of the states. So we have, using that G is uniformly continuous, due to the compactness of its domain, that, for fixed η,

$$G\left(\mu_n(\eta), \frac{n}{N}\right) 1_{\mathcal{H}(n)} = G\left(\sum_{m \in \mathcal{M}} p_n^m(\eta)\, \mu_\infty^0(m, \eta), \frac{n}{N}\right) 1_{\mathcal{H}(n)} + o_{a.s.}(1) \tag{3.21}$$

But, with $\mathbb{P}[\mathcal{H}'] = 1$, this also implies that

$$\begin{aligned}
&\frac{1}{N} \sum_{n=1}^{N} G\left(\mu_n(\eta), \frac{n}{N}\right) 1_{\mathcal{H}(n)} \\
&\quad = \frac{1}{N} \sum_{n=1}^{N} G\left(\sum_{m \in \mathcal{M}} p_n^m(\eta)\, \mu_\infty^0(m, \eta), \frac{n}{N}\right) 1_{\mathcal{H}(n)} + o_{a.s.}(1) \\
&\quad = \frac{1}{N} \sum_{n=1}^{N} G\left(\sum_{m \in \mathcal{M}} p_n^m(\eta)\, \mu_\infty^0(m, \eta), \frac{n}{N}\right) + o_{a.s.}(1)
\end{aligned} \tag{3.22}$$

We can assume that F is uniformly continuous in the first variable, for fixed η in the second variable. So we get from dominated convergence that

$$\lim_{N\uparrow\infty} \mathbb{E}F\left(\frac{1}{N}\sum_{n=1}^{N} G\left(\mu_n(\eta), \frac{n}{N}\right); \eta\right)$$
$$= \lim_{N\uparrow\infty} \mathbb{E}F\left(\frac{1}{N}\sum_{n=1}^{N} G\left(\sum_{m\in\mathcal{M}} p_n^m(\eta)\,\mu_\infty^0(m,\eta), \frac{n}{N}\right); \eta\right) \tag{3.23}$$

Now we use the continuity of G as a function of μ and and the locality of F in the second argument. We can assume that $G(\sum_{m\in\mathcal{M}} p_n^m(\eta)\,\mu_\infty^0(m,\eta), n/N)$ is some function $\tilde{G}(p_n(\eta), n/N; \eta_J)$ with a finite volume J whose size is uniform in n/N. So the dependence of the expression under the expectation on the r.h.s. of the last equation on η is local except for the explicit dependence contained in $p_n(\eta)$. Using now property 2, uniform continuity and dominated convergence to replace η by the independent copy η' (as in the proof of [K1] Proposition 1) finishes the proof of (i).

To prove (ii) we have to show that

$$\lim_{N\uparrow\infty} \mathbb{E}F((\mu_{[t_iN]}(\eta))_{i=1,\dots,k}; \eta)$$
$$= \lim_{N\uparrow\infty} \mathbb{E}'\mathbb{E}F\left(\left(\sum_{m\in\mathcal{M}} p_{[t_iN]}^m(\eta')\,\mu_\infty^0(m,\eta)\right)_{i=1,\dots,k}; \eta\right) \tag{3.24}$$

We can write for the expression under the limit on the l.h.s.

$$\mathbb{E}F((\mu_{[t_iN]}(\eta))_{i=1,\dots,k}; \eta)\, 1_{\bigcap_{i=1,\dots,l}\mathscr{H}([t_iN])} + \mathcal{O}\left(\sum_{i=1,\dots,k} \mathbb{P}[\mathscr{H}([t_iN])]\right)$$
$$= \mathbb{E}F\left(\left(\sum_{m\in\mathcal{M}} p_{[t_iN]}^m(\eta)\,\mu_\infty^0(m,\eta)\right)_{i=1,\dots,k}; \eta\right) 1_{\bigcap_{i=1,\dots,k}\mathscr{H}([t_iN])} + o(1) \tag{3.25}$$

due to the approximation property 1, uniform continuity of F and dominated convergence. The second part of the argument that allows to replace η by η' in the argument of p proceeds again as above. ∎

4. SUPERSTATES IN THE CURIE–WEISS RANDOM FIELD ISING MODEL AND FINITE PATTERN HOPFIELD MODEL

Let us illustrate the concepts that were introduced by our two main examples. We revisit only briefly the

4.1. Curie–Weiss Random Field Ising Model

The model possesses the pure phases $\mu_\infty^{\pm}(\eta) = \mu_\infty^0(\pm m^*, \eta)$ where $m^* = m^*(\beta, \hat{\varepsilon})$ is the largest solution of the averaged mean field equation;

these measures are different for large inverse temperature β and small disorder $\hat{\varepsilon}$.

The form of the superstate can easily be understood. We obtain it if we: 1) assume that the system of size n is in the plus (minus) phase once the sum of the random fields $\sum_{i=1}^{n} \eta_i$ is positive (negative) and 2) replace the process $t \to 1/\sqrt{N} \sum_{i=1}^{[tN]} \eta_i$ by an independent Brownian motion W_t, for large N. More precisely, the following results hold.

Theorem 1. For the Gibbs measure valued paths $(t \mapsto \mu(\eta)_{[tN]})_{0<t\leqslant 1}$ we have the limit in locally conditioned distribution

$$\lim_{N\uparrow\infty} (t \mapsto \mu(\eta)_{[tN]})_{0<t\leqslant 1} =^{\text{lc-law}} (t \mapsto 1_{W_t>0}\mu_{\infty}^{+}(\eta) + 1_{W_t\leqslant 0}\mu_{\infty}^{-}(\eta))_{0<t\leqslant 1} \tag{4.1}$$

in the sense of convergence of finite dimensional marginals or convergence of functions $\int_0^1 dt\, F(\mu_t, t)$ with F continuous on $\mathscr{P}(\Omega)\times[0,1]$. Here W_t is a Brownian motion starting at the origin, independent of η. The variable η is the same on both sides.

Proof. Let us use from [K1] the sets $\mathscr{H}(N) = \{\eta : |\sum_{i=1}^{N} \eta_i| \leqslant N^{(1+\delta)/2}$ and $|\sum_{i=1}^{N} \eta_i| \geqslant N^{\tilde{\delta}}\}$ (for small δ, $\tilde{\delta}>0$) that have $\lim_{N\uparrow\infty} \mathbb{P}[\mathscr{H}(N)] = \mathbb{P}[\mathscr{H}'] = 1$. We showed that the finite volume Gibbs measures satisfy the approximation property 1 with the weights $(1_{\sum_{i=1}^{N}\eta_i>0}, 1_{\sum_{i=1}^{N}\eta_i\leqslant 0})$ (see the proof of Theorem 1, 1′ therein). From this, convergence of the finite dimensional marginals follows directly from Proposition 1 since we have, by the multidimensional Central Limit Theorem, that

$$\frac{1}{\sqrt{N}}\left(\sum_{i=1}^{[t_1 N]} \eta_i, \ldots, \sum_{i=1}^{[t_k N]} \eta_i\right) \xrightarrow{\text{law}} (W_{t_1}, \ldots, W_{t_k}) \qquad \text{with} \quad N\uparrow\infty.$$

To see that convergence holds for functionals $\int_0^1 dt\, F(\mu_t, t)$, we use Proposition 1(i). For continuous $\hat{F}$ we have to verify then that $(1/N) \sum_{n=1}^{N} \hat{F}(1_{\sum_{i=1}^{n}\eta_i>0}, n/N)$ converges in law to $\int_0^1 dt\, \hat{F}(1_{W_t>0}, t)$. But this is easy using e.g. the strong approximation from [Rio] (see also the Hopfield case below) that gives us a Brownian motion W_t on the *same* probability space with the η's s.t. $\sup_{n=1,\ldots,N} |\sum_{i=1}^{n} \eta_i - W_n| = \mathcal{O}_{a.s.}(\log N)$. ∎

Note that the process $t \to 1_{W_t<0}$ has highly singular paths. (Every zero of W_t is limit point of zeros from the right, so $1_{W_t>0}$ is *not* a cadlag[4] function.) Therefor we cannot hope for a much better natural notion of

[4] Meaning: continuous from the right and having left limits.

convergence in this case. Indeed, let us consider the continuous path W'_t that is obtained from the Brownian motion W_t by linear interpolation between integer times. Then the process $1_{W'_{Nt}>0}$, does *not* converge to $1_{W_{Nt}>0}$ with $N \uparrow \infty$ in sup-norm (and also not in the Skorohod-norm that is frequently used for the treatment of jump-processes).[5] This illustrates the point that a suitable notion for the limit in locally conditioned distribution for the paths of Gibbs measures has to depend on the model.

An example where we encounter a nicer and richer structure is provided by

4.2. The Finite Pattern Hopfield Model: Cluster Points of Paths and Superstates

We have to recall general facts about the Hopfield model and some notations from [K1]. For the beautiful results in the case of growing number of patterns that we do not touch here, see [BGP], [BG1]–[BG3], [T]. For $\beta > 1$ there exist precisely $2M$ pure phases that come in pairs of symmetric mixtures

$$\mu^{\nu}_{\infty}(\xi) = \tfrac{1}{2}(\mu^{0}_{\infty}(m^* a^{\nu}, \xi) + \mu^{0}_{\infty}(-m^* a^{\nu}, \xi)) \tag{4.2}$$

Here $m^* = m^*(\beta)$ is the largest solution of the ordinary Curie–Weiss equation $m = \tanh \beta m$ and a^{ν} is the νth unit vector of $\mathbb{R}^M$. To respect this symmetry let us, as in [K1] (by a little change of notation compared to Chapter 3), write $\mathscr{S}$ for the simplex of M-dimensional (and *not* $2M$-dimensional!) probability vectors.

Denoting by $\mathscr{A}$ the $(M(M-1))/2$ dimensional vector space of $M \times M$ symmetric matrices with vanishing diagonal we defined the map $p : \mathscr{A} \to \mathscr{S}$ by

$$p^{\nu}(V) = \frac{\tilde{p}^{\nu}(V)}{\sum_{\mu=1}^{M} \tilde{p}^{\mu}(V)} \qquad \text{where} \quad \tilde{p}^{\nu}(V) = \exp(c(\beta)(V^2)^{\nu\nu}) \tag{4.3}$$

with the constant $c(\beta) = (\beta m^*)/(2(1 - \beta(1 - m^*)^2))$. Let us denote by $W_t = (W_t^{\mu\nu})_{1 \leq \mu, \nu \leq M}$ a Brownian motion with values in $\mathscr{A}$, independent of the disorder variables ξ. It is given by $M(M-1)/2$ independent one dimensional Brownian motions $W_t^{\mu\nu}$ for $\mu < \nu$, and setting $W_t^{\nu\mu} := W_t^{\mu\nu}$ and $W_t^{\mu\mu} := 0$.

[5] One might want to construct "fancy" metrics involving Hausdorff measures that do not see the zero set of the Brownian motion, but we do not pursue this any further in this context.

Then we have

Theorem 2. (i) The cluster points of the sequence of paths $(t \mapsto \mu_{tN}(\xi))_{t \in [\varepsilon, 1]}$ in the uniform metric d_ε are, for almost all pattern ξ, the Gibbs measure-valued paths $(t \mapsto \sum_{\nu=1}^{M} q^\nu(t)\, \mu_\infty^\nu(\xi))_{t \in [\varepsilon, 1]}$ where $(t \mapsto q(t))_{t \in [\varepsilon, 1]}$ is any continuous $\mathscr{S}$-valued function, and $\varepsilon > 0$.

(ii) Let F denote a bounded function on the space of continuous $\mathscr{P}(\Omega)$-valued paths, indexed by $[\varepsilon, 1]$, $\varepsilon > 0$. Assume that F is continuous in the uniform metric d_ε. Then we have the limit in lc-law

$$\lim_{N \uparrow \infty} F(t \mapsto \mu_{tN}(\xi)) =^{\text{lc-law}} F\left(t \mapsto \sum_{\nu=1}^{M} p^\nu \left(\frac{W_t}{\sqrt{t}} \right) \mu_\infty^\nu(\xi) \right) \tag{4.4}$$

Note that the weights on the r.h.s. of the equation (4.4) are singular for $t \downarrow 0$ which forces us to restrict to intervals $[\varepsilon, 1]$ that are bounded against zero. Loosely speaking, the emergence of the Brownian motion explains itself from the approximation of the finite volume free energy differences that are essentially sums of independent random variables, using an invariance principle for multidimensional random walks.

In answer to the question in the little example in the second paragraph of the introduction, we see from (ii) that

$$\begin{aligned} &\lim_{N \uparrow \infty} \sup_{n : [N/2] \leqslant n \leqslant N} \mu_n(\xi)(\sigma_{x_1} \sigma_{x_2}) \\ &\quad =^{\text{lc-law}} (m^*)^2 \sup_{1/2 \leqslant t \leqslant 1} \sum_{\nu=1}^{M} \xi_{x_1}^\nu \xi_{x_2}^\nu p^\nu \left(\frac{W_t}{\sqrt{t}} \right) \end{aligned} \tag{4.5}$$

where the ξ's are the same on both sides. In the case of the above observable, simple convergence of finite dimensional marginals would not have been sufficient to conclude this statement. The a.s. cluster points of this observable are given by the (possibly degenerate) interval $(m^*)^2 [\min_{\nu=1}^{M} \xi_{x_1}^\nu \xi_{x_2}^\nu, \max_{\nu=1}^{M} \xi_{x_1}^\nu \xi_{x_2}^\nu]$.

Proof. In [K1] we saw that the asymptotic form of the weights is governed by the $M \times M$ matrix $b_N(\xi)$, given by $b_N^{\mu\nu}(\xi) = \sum_{i=1}^{N} (\xi_i^\mu \xi_i^\nu - \delta^{\mu\nu})$. It is however even more convenient to use the normal partial sum process $g_N(\xi)$, living on the same probability space, that is a strong approximation in the sense that $\sup_{n=1,2,\dots,N} \|b_n - g_n\| = \mathscr{O}_{a.s.}(\log N)$ (see [Rio], note the typo in [K1]!) It is such that $\gamma_n^{\mu\nu} = \gamma_n^{\nu\mu}$ for $\nu \neq \mu$, $\gamma_n^{\mu\mu} \equiv 0$, $\gamma = (\gamma_n^{\mu\nu})_{1 \leqslant \mu \neq \nu \leqslant M;\, n=1,2,\dots}$ are i.i.d Normal Gaussians (for different $\{\mu, \nu\}$

and n). We write $T_\xi(p)=\sum_{\nu=1}^{M} p^\nu \mu_\infty^\nu(\xi)$. It was shown in [K1] that, for a.e. realization of ξ we had

$$\lim_{N\uparrow\infty} d\left(T_\xi\left(p\left(\frac{b_N(\xi)}{\sqrt{N}}\right)\right), \mu_N(\xi)\right) = \lim_{N\uparrow\infty} d\left(T_\xi\left(p\left(\frac{g_N(\xi)}{\sqrt{N}}\right)\right), \mu_N(\xi)\right) = 0 \tag{4.6}$$

The space of $\mathscr{P}(\Omega)$-valued processes on $[\varepsilon, 1]$ that are continuous w.r.t. d_ε is a Polish space equipped its natural Borel σ-field. From the continuity of the maps $V \mapsto p(V)$ and T_ξ we see that, for fixed ξ, the process $(t \mapsto \sum_{\nu=1}^{M} p^\nu(W_t/\sqrt{t})\, \mu_\infty^\nu(\xi))_{\varepsilon \leqslant t \leqslant 1}$ is the image of the Brownian paths on $[0, 1]$ under a *continuous* (in particular measurable) map; thus it is a proper random element.

We start with the proof of (ii). From Proposition 1(ii) and the r.h.s. of (4.6) follows that convergence in lc-law holds in the sense of convergence of finite dimensional marginals. According to Lemma 1, all that is left is to check the tightness for the asymptotic weights where we conveniently use those given by the Gaussian approximation. Breaking up the N, N'-sup into suprema over sub-blocks of side-length $\delta\bar{N}(1-\varepsilon_1)$ it suffices to show that

$$\lim_{\delta\downarrow 0} \sup_{\bar{N}} \sum_{k=0}^{[(1-\varepsilon_1)/\delta]} \mathbb{P}\left[\sup_{\substack{N, N':\\ (\varepsilon_1+\delta k)\bar{N} \leqslant N < N' \leqslant (\varepsilon_1+\delta(k+1))\bar{N}}} \left\| p\left(\frac{g_N(\xi)}{\sqrt{N}}\right) - p\left(\frac{g_{N'}(\xi)}{\sqrt{N'}}\right)\right\| \geqslant \varepsilon\right] = 0 \tag{4.7}$$

We estimate the probability in the last line uniformly in k. We have from [K1], Lemma 6 that $\|p(V)-p(V')\|_1 \leqslant 4c(\beta)(\|V\|_{ss} + \|V-V'\|_{ss})\, \|V-V'\|_{ss}$ with a certain matrix norm whose form doesn't matter here. It is easy to see from this estimate that (4.7) is implied if we can find a function $R(\delta)$ with $R(\delta)\uparrow\infty$ as $\delta\downarrow 0$ s.t.

$$\lim_{\delta\downarrow 0} \frac{1}{\delta} \sup_N \mathbb{P}\left[\sup_{N' : N \leqslant N' \leqslant (1+\delta)N} \left\|\frac{g_N}{\sqrt{N}} - \frac{g_{N'}}{\sqrt{N'}}\right\| \geqslant \frac{\varepsilon'}{R_\delta}\right] = 0 \tag{4.8}$$

and

$$\lim_{\delta\downarrow 0} \frac{1}{\delta} \sup_{N=1,2,\ldots} \mathbb{P}\left[\left\|\frac{g_N}{\sqrt{N}}\right\| \geqslant R_\delta\right] = 0 \tag{4.9}$$

for any small enough $\varepsilon' > 0$. The last probability does not even depend on N and is well known to be bounded above by an exponential $Const\, e^{-const\, R_\delta^2}$. To make the limit zero we might thus choose e.g., $R_\delta = \log 1/\delta$. To show (4.8) write

$$\sup_{N' : N \leqslant N' \leqslant (1+\delta) N} \left\| \frac{g_N}{\sqrt{N}} - \frac{g_{N'}}{\sqrt{N'}} \right\| \leqslant \left(1 - \frac{1}{\sqrt{1+\delta}}\right) \left\| \frac{g_N}{\sqrt{N}} \right\| + \frac{1}{\sqrt{N}} \sup_{N' : N \leqslant N' \leqslant (1+\delta) N} \left\| \sum_{n=N+1}^{N'} \gamma_n \right\| \tag{4.10}$$

Since $(1 - 1/(\sqrt{1+\delta}))\, R_\delta \to 0$ we see from (4.9) that the first term on the r.h.s. can be ignored. To treat the second term on the r.h.s. we can consider the matrix norm that is given by the sum of the modulus of the elements. Using the maximal inequality $\mathbb{P}[\max_{1 \leqslant j \leqslant L} |\sum_{i=1}^{j} \gamma_i^{\mu\nu}| \geqslant x] \leqslant 2e^{-(x^2/2L)}$ gives then

$$\frac{1}{\delta} \sup_N \mathbb{P}\left[\frac{1}{\sqrt{N}} \sup_{N' : N \leqslant N' \leqslant (1+\delta) N} \left\| \sum_{n=N+1}^{N'} \gamma_n \right\| \geqslant \frac{\varepsilon'}{R_\delta} \right] \leqslant \frac{e^{-const\, \varepsilon'^2/(R_\delta^2 \delta)}}{\delta} \to 0 \tag{4.11}$$

with $\delta \downarrow 0$, which finishes the proof of (ii).

The proof of (i) follows from the following probabilistic fact.

Lemma 2. Denote by $(V_t)_{t \in \mathbb{R}_\geqslant}$ the process obtained from a d-dimensional random walk with standard Gaussian increments by linear interpolation for non-integer t. Let $\phi : (0, 1] \to \mathbb{R}^d$ be continuous, $\varepsilon > 0$ and $\delta > 0$. Then

$$\mathbb{P}\left[\sup_{\varepsilon \leqslant t \leqslant 1} \left\| \frac{V_{tN}}{\sqrt{N}} - \phi(t) \right\| < \delta \text{ for inf. many } N \right] = 1 \tag{4.12}$$

Here is a short proof for the convenience of the reader: Consider blocks indexed by k with endpoints $N_k = k!$. It suffices to show that, for each $\delta_1 > 0$ we have

$$\mathbb{P}\left[\sup_{\varepsilon \leqslant t \leqslant 1} \left\| \frac{V_{tN_k} - V_{tN_{k-1}}}{\sqrt{N_k - N_{k-1}}} - \phi(t) \right\| < \delta_1 \text{ for inf. many } k \right] = 1 \tag{4.13}$$

In fact, assuming this, we have from the Law of Iterated logarithm that,

$$\begin{aligned}
&\sup_{\varepsilon \leqslant t \leqslant 1} \left\| \frac{V_{tN_k}}{\sqrt{N_k}} - \phi(t) \right\| \\
&\leqslant \sup_{\varepsilon \leqslant t \leqslant 1} \left\| \frac{V_{tN_{k-1}}}{\sqrt{N_k}} \right\| + \sqrt{1-\frac{1}{k}} \sup_{\varepsilon \leqslant t \leqslant 1} \left\| \frac{V_{tN_k} - V_{tN_{k-1}}}{\sqrt{N_k - N_{k-1}}} - \phi(t) \right\| \\
&\quad + \left(1 - \sqrt{1-\frac{1}{k}}\right) \sup_{\varepsilon \leqslant t \leqslant 1} \|\phi(t)\| \\
&\leqslant \frac{Const \sqrt{N_{k-1} \log\log N_{k-1}}}{\sqrt{N_k}} + \sqrt{1-\frac{1}{k}}\, \delta_1 \\
&\quad + \left(1 - \sqrt{1-\frac{1}{k}}\right) Const \leqslant 2\delta_1 \qquad (4.14)
\end{aligned}$$

for infinitely many (large enough) k. By Borel–Cantelli, it suffices to show now for the "scale invariant" expression in (4.13) that $\lim\inf_N \mathbb{P}[\sup_{\varepsilon \leqslant t \leqslant 1} \|(V_{Nt}/\sqrt{N}) - \phi(t)\| < \delta_1]$ is strictly positive. But this is easy to see going back to finite dimensional approximations, using that $\phi(t)$ is uniformly continuous on $[\varepsilon, 1]$, and using the maximal inequality to control the fluctuations. ▮

Using the explicit map from an M-dimensional sub-manifold of $\mathbb{R}(M(M-1))/2$ to $\mathbb{R}^M$ that was constructed in the proof of [K1] Lemma 8 it is not difficult to see that $V \mapsto p(V)$ is not only onto $\mathscr{S}$ but that the closure of the image of the continuous functions $(t \mapsto V_t)_{\varepsilon \leqslant t \leqslant 1}$ under the map p are the continuous functions $(t \mapsto p(t))_{\varepsilon \leqslant t \leqslant 1}$. From this statement, the right equality of the approximation (17), Lemma 2, and continuity of $V \mapsto p(V)$ we obtain (i). ▮

The same type of arguments also show that the *cluster points* of the sequence of *empirical metastates* $\kappa_N(\xi)$ are, a.s., the images of *all* probability measures on the space of weights $\mathscr{S}$ under the map $q \mapsto \sum_{\nu=1}^{M} q^\nu \mu_\infty^\nu(\xi)$. (Here we have to use again that $V \mapsto p(V)$ is onto and that we can approximate any measure on $\mathscr{S}$ by a measure of the form $\int_0^1 dt\, \delta_{p(\phi(t))}$ with continuous ϕ.)

For the lc-limit we obtain $\lim_{N\uparrow\infty} \kappa_N(\xi) =^{\text{law}} \int_0^1 dt\, \delta_{\sum_{\nu=1}^{M} p^\nu(W_t/\sqrt{t})\mu_\infty^\nu(\xi)}$ with ξ being the same on both sides.

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